Canadian Board of Examiners for Professional Surveyors
Core Syllabus Item
C 2: LEAST-SQUARES ESTIMATION AND DATA ANALYSIS

Study Guide:

1. Apply Knowledge of matrix theory, statistics and estimation:
   - conduct manipulation of matrix algebra involved in adjustment of observations,
   - linearize a non-linear system,
   - apply knowledge of probability and statistics, and
   - demonstrate an understanding of the principles of least square estimation and properties.

Sample Questions:

Q1.1. Define and explain briefly the following terms:

a. Expectation
b. Unbiasedness of an estimator
c. Standard deviation
d. Mean value
e. Variance
f. Root mean square error
g. Precision
h. Accuracy
i. Redundancy of a linear system

Q1.2. Explain the difference between:

a. Accuracy and precision
b. Standard deviation and root mean square error

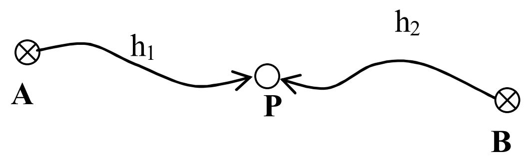
Q1.3. What are the advantages of using least squares method?

See related chapters in "Analysis and Adjustment of Survey Measurements" by Mikhail and Gracie (1981).

2. Analyze measurement errors and modelling, perform random error propagation and pre-analysis of survey measurements:
   - demonstrate an understanding different types of errors and their characteristics,
   - demonstrate an understanding different types of models and characteristics,
   - apply law of random error propagation to determine variance and covariance matrix, and
   - conduct pre-analysis of survey measurements.

Sample Questions:

Q2.1. Given a leveling network below where A and B are known points, h_1 and h_2 are two height difference measurements with standard deviation of σ_1 and σ_2, respectively and $\sigma_1 = 2\ \sigma_2$. Determine the value of σ_1 and σ_2 so that the standard deviation of the height solution at P using least squares adjustment is equal to 2mm.



Answer:

$\sigma_1 = 2\sqrt{5}\ \text{mm}, \sigma_2 = \sqrt{5}\ \text{mm}$

Q2.2. Given the variance-covariance matrix of the measurement vector $\ell = \begin{bmatrix} \ell_1 \\ \ell_2 \end{bmatrix}$:

$$C_\ell = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} \end{bmatrix}$$

and two functions of $\ell : x = \ell_1 + \ell_2$ and $y = 3\ell_1$, determine $C_{xy}, C_{x\ell}, C_{y\ell}$

Answer:

$C_{xy} = 3$, $C_{x\ell} = (1 \quad 1)$, $C_{y\ell} = (2 \quad 1)$

Q2.3. Question 3: Explain the difference between random and systematic errors

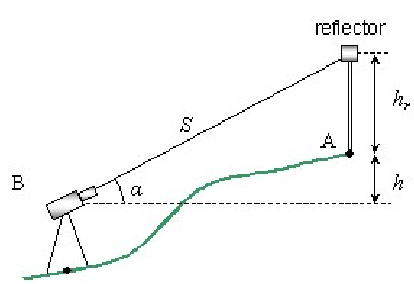
Q2.4. A surveyor's error (one sigma) due only to reading is determined to be 1.5" when making observations with a particular instrument. After repeatedly pointing on a distant target with the same instrument, the surveyor determines the combined error due to both reading and pointing to be 2.6". What is the surveyor's pointing error?

Answer:

2.1"

Q2.5. The height of a survey station (A) above the instrument at (B) is required with an accuracy of 0.01m from the measurement of the slope distance s, the vertical angle α and the target height h_r.

$h = s\sin(\alpha) - h_r$



a. Estimate $\sigma_s, \sigma_\alpha, \sigma_{hr}$ assuming equal accuracies and with approximate values of $s = 400m$, $\alpha = 30^o$.

b. If σ_α is limited by the instrument used to be 5.0" for example, re-evaluation σ_s, σ_{hr} to accommodate for this limitation in σ_α.

Answer:

$\sigma_s = 0.0115m, \sigma_\alpha = 3.4"$, $\sigma_{hr} = 0.006m$

$\sigma_s = 0.008m$, $\sigma_{hr} = 0.004m$

See Chapter 2, 6 and 7 of "Analysis and Adjustment of Survey Measurements" by Mikhail and Gracie (1981)

3. Formulate least squares adjustment problems (condition, parametric and combined cases):

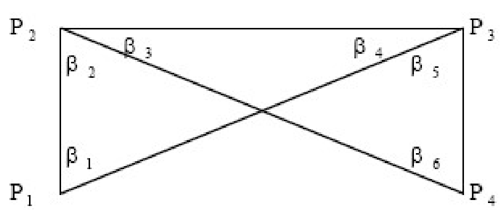
- formulate parametric adjustment models (functional and stochastic),
- formulate condition adjustment models (functional and stochastic), and
- formulate combined adjustment models (functional and stochastic).

Sample Questions:

Q3.1. Explain the difference between functional and stochastic models

Q3.2. What is the criterion of the least squares method?

Q3.3. The angles shown in the following figure are measured with a theodolite and their observed values and standard deviations are listed in the table. Formulate the functional and stochastic models to determine the adjusted values for these angles using the condition approach.



Angle	Observed values	Standard Deviation
β_1	44° 50' 44"	2.6"
β_2	46° 10' 25"	1.5"
β_3	45° 55' 12"	1.5"
β_4	43° 04' 03"	1.5"
β_5	48° 32' 45"	1.5"
β_6	42° 27' 42"	2.6"

Answer:

Angle	Adjusted values

β_1	44° 50' 27"
β_2	46° 10' 19"
β_3	45° 55' 11"
β_4	43° 04' 02"
β_5	48° 32' 50"
β_6	42° 27' 57"

Q1.1. Assume that the points P1 and P2 in Question 3 are two control points whose known coordinates are provided in the following table. Formulate the functional and stochastic models to determine the coordinates of the points P3 and P4 using parametric approach.

Control points	**X (m)**	**X (m)**
P1	0.0	0.0
P2	1000.0	0.0

See Chapters 3, 4 and 9 of "Analysis and Adjustment of Survey Measurements" by Mikhail and Gracie (1981))

4. Derive adjustment equations of different cases and conduct least square adjustment for geomatics problems such as levelling, traverse, triangulation and trilateration networks:
   - derive parametric adjustment equations,
   - derive condition adjustment equations,
   - derive combined adjustment equations, and
   - apply to geomatics problems such as levelling, traverse, triangulation and trilateration networks.

Sample Questions:

Q4.1. Given the following mathematical model

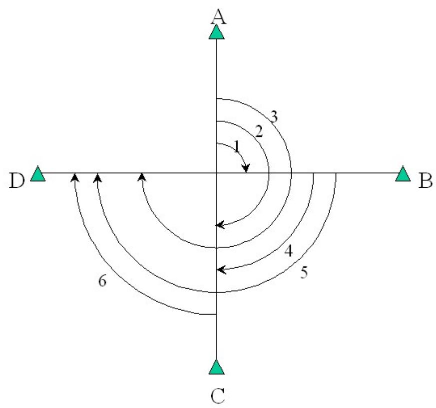
$$f(\ell, x) = 0 \qquad C_\ell \quad C_x$$

where f is the vector of mathematical models, x is the vector of unknown parameters and C_x is its variance matrix, ℓ is the vector of observations and C_ℓ is its variance matrix.

a. Linearize the mathematical model
b. Formulate the minimization function
c. Derive the least squares normal equation
d. Derive the least squares solution of the unknown parameters

Q4.2. Conduct condition adjustment to estimate x using angle observations l measured in the same accuracy of $\sigma = 1''$ and given in the table below.

$$x^T = \begin{bmatrix} \alpha_1 \\ \alpha_4 \\ \alpha_6 \end{bmatrix} \qquad l^T = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \\ \alpha_6 \end{bmatrix}$$



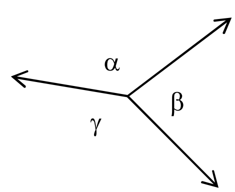
Angle	Observations
α_1	89°59’58.3’’
α_2	180°00’01.4’’
α_3	270°00’00.2’’
α_4	89°59’59.8’’
α_5	179°59’57.0’’
α_6	90°00’03.1’’

Answer:

Angle	**Adjusted values**
α_1	90° 00’ 0.35”
α_4	89° 59’ 59.15”
α_6	90° 00’ 0.55”

Q4.3. Given the angle measurements at a station along with their standard deviations, apply the parametric least squares adjustment to the problem.

Angle	**Measurement**	**Standard Deviation**
α	134°38’56”	6.7”
β	83°17’35”	9.9”
γ	142°03’14”	4.3”



Answer:

Angle	**Adjusted values**
α	134°39’0.2”
β	83°17’44.1”

γ	142º03'15.7"

See Chapters 3, 4 and 9 of "Analysis and Adjustment of Survey Measurements" by Mikhail and Gracie (1981))

5. Assess the quality of the adjustment solutions (variance factor, variance-covariance matrix, error ellipse):
   - estimate the variance factor,
   - determine variance-covariance matrix of parameters obtained from least square adjustment, and
   - demonstrate an understanding the concept of error ellipse and determine its major axes and orientation.

Sample Questions:

Q5.1. Question 1: Given the variance-covariance matrix of the horizontal coordinates (x, y) of a survey station, determine the semi-major, semi-minor axis and the orientation of the standard error ellipse associated with this station.

$$C_x = \sigma_0^2 \begin{bmatrix} 0.380 & 0.025 \\ 0.025 & 0.510 \end{bmatrix}$$

where σ_0 = 2cm.

Answer:

semimajor =1.43cm, semiminor = 1.23cm, orientation = -10o31'08"

Q5.2. Given the angle measurements in a plane triangle along with their weight, calculate the variance factor, the variance-covariance matrix and the standard deviations of the adjusted angles.

Angle	Measurement	Weight
ℓ_1	41º33'45"	1.00
ℓ_2	78º57'55"	0.67
ℓ_3	59º27'50"	0.50

Answer:

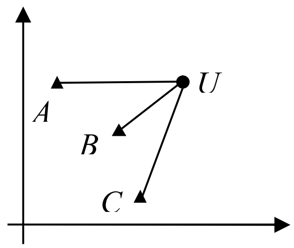
variance factor is $14.15''$, $C_{\hat{\ell}} = \begin{bmatrix} 155.74 & -66.56 & -89.18 \\ -66.56 & 199.67 & -133.11 \\ -89.18 & -133.11 & 222.30 \end{bmatrix}$,

$\sigma_{\hat{\ell}_1} = 12.50''$ $\sigma_{\hat{\ell}_2} = 14.13''$ $\sigma_{\hat{\ell}_3} = 14.91''$

Q5.3. In a trilateration network shown below, U is an unknown point and A, B and C are three control points with coordinates of

Control Points	A	B	C
X (m)	865.40	2432.55	2865.22

Y (m)	4527.15	2047.25	27.15



Assume that the distances d_{AU}, d_{BU}, and d_{CU} are measured with the same precision and their observed values are 6049.00m, 4736.83m and 5446.29m, respectively.

a. Estimate the coordinates of the point U and its variance-covariance matrix

b. Calculate the semi-major, semi-minor axis and the orientation of the standard error ellipse for points U

c. Calculate the variance factor.

Answer:

a. X_U=6861.32m, Y_U=3727.82m; $C_x = \begin{bmatrix} 0.0063 & -0.0073 \\ -0.0073 & 0.0249 \end{bmatrix}$

b. semimajor = 0.1656m, semiminor = 0.0615m, orientation = 109°03'54"

c. 0.01m

See Chapters 6 and 8 of "Analysis and Adjustment of Survey Measurements" by Mikhail and Gracie (1981)

6. Perform statistical tests on mean and variance to detect and identify outliers in observations (normal, Chi-square, t Student and F distributions, statistical hypotheses, type I and II errors):

- perform statistical tests on mean and variance to detect and identify outliers in observations,
- determine the confidence interval of adjusted parameters,
- select appropriate testing methods (normal, Chi-square, t Student and F distributions), and
- determine the confidence level and error probability of statistical decisions (significance level, test power, type I and II errors).

Sample Questions:

Q6.1. Define or explain the following terms:

a. Null hypothesis and alternative hypothesis

b. Correlation coefficient

c. Type I and type II errors in statistical testing

Q6.2. An angle is measured 10 times. Each measurement is independent and made with the same precision. The sample standard deviation is s = 7.3". Test at a significance level of 5% the hypothesis that the population standard deviation σ of the measurements is 2.0" against the alternative that σ is not 2.0".

Answer:

σ = 2.0" is rejected at 5% level of significance

Q6.3. A baseline of calibrated length (μ) 1153.00m is measured 5 times. Each measurement is independent and made with the same precision. The sample mean ($\bar{x}$) and sample standard deviation (s) are calculated from the measurements:

$\bar{x}$ = 1153.39m s = 0.06m

a. Describe the major steps to test the mean value.

b. Test at the 10% level of confidence if the measured distance is significantly different from the calibrated distance.

Answer:

b. the measured distance is significantly different from the calibrated distance at 10% level of confidence

See Chapter 8 of "Analysis and Adjustment of Survey Measurements" by Mikhail and Gracie (1981)